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Learning Mathematics From Examples and by Doing

Xinming Zhu
Chinese Academy of Sciences

Herbert A. Simon
Carnegie-Mellon University

This article demonstrates the feasibility and effectiveness of teaching several mathematical skills by presenting students with carefully chosen sequences of worked-out examples and problems—without lectures or other direct instruction. Thinking-aloud protocols of 20 students learning factorization by this method are analyzed to determine the kinds and depth of understanding students attained. Students did not simply memorize procedures but were able to recognize when the procedures were applicable and to apply them. Most students were also able to use their understanding of the concept of factorization to help learn the procedures and to check their results. The method of learning from examples has now been tested successfully with a class covering the entire 3-year curriculum in algebra and geometry in a Chinese middle school.

Human learning takes place in a wide variety of situations and almost surely employs many different processes. During the past few years, considerable attention has been paid to the ways in which people can learn procedures by examining worked-out examples and by solving problems—learning from examples and learning by doing (e.g., Anzai, 1978; Anzai & Simon, 1979; Neves, 1978; Neves & Anderson, 1981; Sweller & Cooper, 1985). Moreover, it has been shown that such learning can be simulated using computer programs known as *adaptive production systems;* these computer programs now provide a model for the same processes in human learning.

In this article, we discuss the process of learning from examples, examining several experiments with high-school students employing this

Requests for reprints should be sent to Herbert A. Simon, Department of Psychology, Carnegie-Mellon University, Pittsburgh, PA 15213.

learning paradigm. In addition to permitting comparison of this learning strategy with more traditional ones, the experiments provided information about the processes actually used by students who employed the new strategies. By "traditional strategies," we mean procedures in which the teacher and textbook play an active role in presenting and explaining the material to be learned.

Our interest in the prospects of learning from examples was sparked by incidents such as this: A student who was late for class missed the teacher's lecture, but at the end of the class, he looked at the problems worked by another student. When the tardy student was tested, we were surprised to see that he worked the test problems correctly. Apparently he had learned by studying the worked-out examples. How generalizable is the result? How efficient is the process?

SOME HISTORICAL COMMENTS

Of course, the idea of learning from examples is not new. Worked-out examples constitute an important element in textbook presentations of new concepts and procedures. These examples can be shown to contain enough information about the procedures to permit diligent students to learn them without additional instruction, a fact many good students appear to know and exploit. Presumably, the students infer from the examples the essential procedures and then internalize them so that they can apply these procedures to new problems.

Closely related to learning from examples is the alternative process of learning by doing—that is, learning by solving problems. If feedback is provided on whether an answer is correct, and if the problem solver persists until the problem is solved, then the solution path (however halting and inefficient the search for it may have been) provides a worked-out example. This example can be used as a basis for finding the correct algorithm in the same manner that a worked-out example provided by the teacher can be used.

In the literature of education, the ideas of learning from examples and learning by doing also have a long history. They have their roots in ideas about child-centered education associated with the names of Pestalozzi, Froebel, and Herbart and in the role assigned to the learner's experiences in John Dewey's prescriptions for progressive education. Several followers of Dewey and Piaget have been strong advocates of discovery learning, and there has been a significant amount of research on the conditions under which discovery learning is, or is not, likely to be effective and efficient (Shulman & Keislar, 1966). Discovery learning is, of course, nearly synonymous with learning by doing.

Psychological research on concept attainment (Bruner, Goodnow, & Austin, 1956), and computer modeling of concept attainment processes (Hovland & Hunt, 1960; Hunt, 1962) have always used a learning-from-examples paradigm, which is perhaps not surprising because the interest lay in inductive discovery rather than in learning per se. In a different domain, Siklóssy (1972) showed that the vocabulary and grammar of languages could be learned by a computer program from examples of sentences paired with semantic descriptions ("pictures") of the sentence meanings.

Psychologists interested in creative processes have also long observed the common element of discovery that is central to both creativity and learning by doing. In what is perhaps the earliest treatment of creativity from an information processing point of view (Newell, Shaw, & Simon, 1958/1979), a computer learning program that learned by doing was described, and the importance of *hindsight* (analysis of problem solutions after discovery) was underscored:

> The learning programs we have mentioned have two important characteristics in common: (1) they consist in a gradual accumulation of selective principles that modify the sequence in which possible solutions will be examined in the problem space; (2) the selective principles are obtained by hindsight—by analysis of the program's successes in its previous problem solving efforts. (Newell et al., 1958/1979, p. 171)

Research on learning within an information processing paradigm received a new impetus about 15 years ago with Waterman's (1970, 1975) invention of *adaptive production systems* as a programming model of the learning mechanism. An adaptive production system is simply a computer program capable of modifying itself adaptively by constructing new instructions (productions) and adding them to its memory. The productions provide a detailed model of what must be learned to perform the task, and the adaptive productive system provides an explanation of how the productions that make up this model can be acquired successively.

Adaptive production systems provided an operational method for combining task analysis (e.g., Gagné, 1965; Gagné & Briggs, 1974; Resnick, 1976) with a specific learning theory—hence, for constructing an orderly framework within which materials for learning from examples and learning by doing could be designed and instructional materials could be sequenced.

Neves (1978) was the first to apply these ideas to school learning with an adaptive production system that was able to learn to solve linear algebraic equations. Considerable research activity has followed (e.g., Anzai, 1978, Anzai & Simon, 1979; Neves & Anderson, 1981), and these learning theories are now being used by Anderson (1987) and others in the design of tutoring systems. There have also been new beginnings of experimentation with learning from examples and learning by doing (e.g., Sweller & Cooper, 1985). Our own empirical research on this topic began in 1983.

SCOPE OF THIS ARTICLE

We have carried out several experiments to test the feasibility and efficiency of learning from examples and learning by doing. In the first experiment, we took verbal protocols from some of the participants to learn more about the processes they were using. In the remaining experiments, we took protocols only for the geometry task, but we measured the speed and extent of learning in comparison with the learning of students who were taught the same material conventionally.

This article focuses primarily on the protocol material and what it discloses about the processes students use to learn from worked-out examples and by solving problems. Our main interest was in determining whether, and with what degree of understanding, schoolchildren can learn school subjects by these methods. Since we are concerned mainly with the nature and feasibility of the methods, we present some preliminary findings on their efficiency in comparison with traditional methods only briefly, postponing to another occasion a detailed report on outcomes. In addition to elucidating the learning processes that appear in the protocols, we pay a good deal of attention to the question of whether the methods of learning from examples and learning by doing encourage rote learning or whether they promote, instead, learning with understanding.

INSTRUCTIONAL MATERIALS

Experiments on learning from examples and learning by doing were carried out with learning materials for several tasks contained in the mathematics curriculum of fifth-, seventh-, and eighth-grade students. The tasks included simplifying fractions, factoring quadratic expressions, manipulating terms with exponents, and solving several geometry problems.

Training Materials

The preparation of training materials was preceded by a careful and detailed analysis of each task. On the basis of this analysis, a production system capable of performing the task was constructed to represent the skills that students would acquire in mastering the task. We assumed that a person able to perform the task possesses a set of productions very similar to those in the model.

The training materials were then designed to motivate learning the productions successively. That is, the examples and problems were sequenced so that the initial problems could be handled with a small subset of the productions, and subsequent problems required additional productions for their solution. Thus, in accordance with the usual principles for shaping behavior, learners could attend to one or a few aspects of the problem situation at a time.

A Production System

The first section of the Appendix sets forth a production system for factoring quadratic expressions of the form $x^2 + ax + b$ in which a and b are positive or negative integers. We used this production system to construct our examples and problems for teaching this skill. The procedure begins by finding all pairs of positive integers that factor the constant term b of the quadratic. Then it divides into cases according to the signs of the linear and constant terms. Depending on these signs, a pair of factors is selected whose sum (if the sign of the constant term is positive) or difference (if the sign of the constant term is negative) is equal in absolute magnitude to the coefficient of the linear term of the quadratic. Signs are now assigned to the factors depending on the arrangement of signs in the quadratic. If the constant term is positive, then both factors are positive (negative) as the coefficient of the linear term is positive (negative), respectively. If the constant term is negative, then the larger factor is positive (negative) as the coefficient of the linear term is positive (negative), respectively; and the smaller factor has a sign opposite that of the larger factor.

Fortunately for the feasibility of students' learning to factor, these rules are easier to understand in the context of examples and problems than they are when thus stated in prose, or when put in the production system format of the Appendix. Moreover, the principle underlying the factoring of quadratics can be explained, as we see later, in a unified way, without invoking four special cases that depend on the signs of the coefficients. Nevertheless, factoring is usually taught in terms of the four cases, and we think this is probably an effective procedure for focusing the students' attention on one process at a time. First, they learn to search for a pair of factors of the constant term adding up to the coefficient of the linear term. When they have acquired this skill, they focus on the signs of the linear and constant terms of the quadratic and on the effects of these signs on the selection of the correct factor pair and the assignment of signs to the factors.

The details of the particular production system described in the Appendix are probably not important, for the task could be accomplished about equally well with several alternative systems. In fact, the protocol evidence shows that the production systems acquired by our students are not identical but are variants of this basic scheme.

The alternative to factoring by recognizing distinct cases for different combinations of the signs of terms in the quadratic is to factor "algebraically," that is, to use signed numbers throughout. Then the rule for factoring a quadratic reduces to a single case: Find all pairs of factors (regarding $+a$ and $-a$ as distinct factors) of the (signed) constant term of the quadratic; then select the pair whose algebraic sum equals the (signed) coefficient of the linear term, and write the factorization as $(x + c)(x + d)$, in which c and d are the (signed) factors—$x^2 + ax + b = (x + c)(x + d)$, in which $c \times d = b$, $c + d = a$.

As we shall see, there is evidence that many of our participants learned the more general principle, even if they used the productions for the four specific cases in their actual problem solving.

Prerequisite Knowledge

To learn to factor quadratic expressions, students must have some prerequisite knowledge (see Appendix: 2. Review Knowledge). In particular, they must know what factoring means. Before reaching quadratics in the curriculum, they have already studied the factoring of integers and of monomials in algebraic expressions. We convey additional information about quadratics to them by defining the factoring of a quadratic expression into two linear expressions as the inverse of the operation of multiplying the two linear expressions to form the quadratic. Because the students already have the skill of multiplying algebraic expressions, they should understand this definition of the task of factoring; this definition can also provide them with a means for checking (by multiplying back) whether they have factored correctly. Hence, in learning either from examples or by doing, they can obtain feedback on the correctness of their answers. This feedback provides essential information for the learning process.

From the definition of factoring as the inverse of multiplication and from the worked-out examples of multiplication in the training materials, students can derive the "algebraic" algorithm for factoring that we have already presented. They can observe that the constant term in a quadratic is the product of the factors, while the coefficient of the linear term is the (algebraic) sum of the factors. To facilitate their noticing these relations, we include in the review knowledge several problems of factoring integers, noting the sums of the factors. We also carry out the multiplication in steps so that the summation of factors to form the coefficient of the linear term of the quadratic is made explicit.

Awareness of these relations can provide guidance to the students' search for solutions in the learning-by-doing condition as well as in the learning-from-examples condition. As we have seen, the laws of signs in factoring, which we have represented by separate productions, are also implicit in the definition of factoring. In principle, students should be able to infer the laws of signs from the general definition, hence finding a rationale for each production embodying these laws. Many of our subjects did accomplish this.

In presenting the prerequisite knowledge prior to the learning trials, we go a little further than simply defining factoring. We also give the students practice in finding all pairs of integral factors of a positive integer and practice in finding the sum of each pair. In presenting the product of two linear expressions, we could also have explicitly called attention to the fact that the linear term of the quadratic is the algebraic sum of the two constant terms of the linear expressions, while the constant term of the quadratic is the algebraic product of the two constant terms of those expressions. In fact, it did not prove necessary to provide this additional review knowledge for students to acquire the requisite skills from the examples and problems.

EXPERIMENT 1

Method

The task was to learn to factor quadratic algebraic expressions of the form $x^2 + ax + b$, in which a and b are positive or negative integers and the factors are also integers. In these experiments, we consider only expressions in which the coefficient of the quadratic term is $+1$.

Two groups participated in the experiment: (a) 20 students who provided verbal protocols while they were run individually and (b) two classes, totaling 98 students, who were run in the classroom setting without taking verbal protocols. The protocol group provides us with rich data about the processes they used. Half worked in a learning-from-examples condition, and half worked in a learning-by-doing condition. (One of the learning-by-doing protocols was lost, so our protocol analysis for this condition contains only 9 students.)

The classroom group provided us with a larger sample of data on learning speeds and levels of learning and worked in the learning-from-examples condition. In this initial experiment, no comparison was made with a control group using conventional learning methods. Our initial goal was to determine whether learning by example or by doing was feasible for a standard algebra skill and to gain an understanding of the learning process.

The students were approximately 13 years old and were enrolled in the first algebra course in a middle school in Beijing, China. The school is near the middle range of Beijing schools in terms of student ability. In the standard algebra curriculum, two class sessions are usually devoted to teaching the factoring of quadratics, and homework is assigned for each evening.

The procedures were as follows (see Appendix):

1. Take pretest.
2. Review knowledge prerequisite for the task.
3. Take second pretest (five problems).
4. Learn from examples or by doing.
5. Take final test (identical with second pretest).

Results

Classroom Group

We begin with a brief report on the performance of students who learned in the classroom without providing protocols. Because 4 classroom students could solve the pretest problems, their performance was excluded from the data we analyzed. Of the remaining 94, 78 (83%) solved the test problems correctly after reviewing the prerequisite knowledge and working through a few examples and problems; 8 (8.5%) solved only three of the five test problems correctly; and 8 (8.5%) solved two or fewer problems. These results were achieved within the time of a single class session, demonstrating that learning this skill from examples is wholly feasible within the time allotted in the curriculum.

Protocol Group

Students in the protocol group were run individually and were asked to talk aloud while they were working. Their protocols were tape-recorded and transcribed; for the purpose of this article, they have been translated literally into English. Ten protocol students were provided with materials that included worked-out examples; the other 10 were provided only with a carefully arranged sequence of problems to be solved (Step 4 on the abovementioned list). None of the students could solve any of the pretest problems (Steps 1 and 3) or build appropriate procedures. After spending about 25 min working through examples and doing problems, all 20 experimental participants could solve the test problems (Step 5), and 15 of

TABLE 1
Protocol of E1 on Part I of
Learning From Examples

[Student looking at first exercise]

1. According to the example,
2. Exercise 1, $x^2 + 11x + 18$ is equal to x . . .
3. [Looking at Example 1] $2 + 3$ is equal to the coefficient 5,
4. 2×3 is equal to the constant 6.
5. [Looking at Example 2] This example is $1 + 6 = 7$,
6. $1 \times 6 = 6$.
7. Exercise 1 [$x^2 + 11x + 18$] is that $x + 3$ multiplies $x + 6$.
8. Exercise 2 [$x^2 + 9x + 18$] is that $x + 6$ multiplies $x + 3$.
9. Exercise 3 [$x^2 + 19x + 18$] is that $(x + 9)(x + 2)$.

[Reviewing Exercise 1]

10. That is wrong.
11. $3 + 6$ is not equal to 11.
12. Exercise 1 is $x + 2$ multiplying $x + 9$,
13. $2 + 9 = 11$
14. $2 \times 9 = 18$
15. This one [Exercise 2] $3 + 6 = 9$,
16. $3 \times 6 = 18$.
17. This one [Exercise 3] is also wrong.
18. It should be $x + 1$ multiplies $x + 18$,
19. $1 + 18 = 19$.
20. $1 \times 18 = 18$.

Note. For the problems, see Appendix, 4A, Part I.

the 20 answered the questions about the procedures correctly. In the following paragraphs, we examine their protocols in more detail.

Protocols: Part I

From the protocols, we obtain a rather clear picture of how the participants learn from examples. We begin by discussing the protocols of two students on Part I of the materials: E1, who was learning from examples (E condition); and D7, who was learning by solving problems (D condition). Their protocols on Part I are shown in Tables 1 and 2, respectively. In the following discussion, numbers in parentheses refer to lines in the protocol sequence.

E1's learning. E1 begins by looking at the first exercise (2), then looks back (3 to 6) at the first two of the five examples provided, and checks that the sum of the numerical factors equals the coefficient of the linear term of the quadratic and that the product of the factors equals the constant term. This student then proceeds to solve Exercises 1 to 3 (7 to 9) but pays attention only to the second of these two conditions, thereby obtaining

TABLE 2
Protocol of D7 on Part I of
Learning by Doing

[Student is doing the first exercise]

1. [Exercise 1] $x^2 + 5x + 6 = (x + \quad)(x + \quad)$
2. equals $(x + 11)(x + 30)$.
3. According to multiplication of polynomials, $(x + 11)(x + 30)$ . . .
4. their product . . .
5. That is wrong.
6. It should be $(x + 2)$ multiplied by $(x + 3)$.
7. Since $x \cdot x = x^2$
8. $2 \cdot x = 2x$
9. $3 \cdot x = 3x$
10. $2x + 3x = 5x$
11. 2 multiplied by 3 equals 6.
12. [Experimenter] What rule have you found?
13. I found a rule.
14. The numbers in these brackets . . .
    [the constants of the two factors]
15. added to each other, multiplied by x, we'll get this number
    [the coefficient of the linear term].
16. These two figures multiplied by each other will equal the
    number at the end [the constant].

Note. For problems, see Appendix, 4B, Part I.

wrong answers. He checks the first condition for Exercise 1 and sees it is not satisfied (10 to 11). He now (12 to 14) finds a pair of factors satisfying both conditions. Then he goes back (15 to 20) and corrects Exercises 2 and 3. By the time he has finished, he has essentially built the production rule for positive coefficients.

The examples enable E1 to review the conditions—learned previously from multiplication of linear terms—that the constant term in the quadratic equals the product of the constants in the linear terms (the factors), while the coefficient of the linear term in the quadratic equals the sums of these factors. However, possibly due to limits of attention span, he constructs erroneous solutions that take into account only the former of the two requirements. His understanding that factoring is the inverse of multiplication allows him to check his results and to discover that they are wrong.

From the protocol, we see clearly how E1 uses previously acquired knowledge both to generate his solutions and to test them. We also see the difficulty he encounters in using all his knowledge. He initially generates only one pair of factors that satisfies the product condition without considering whether it satisfies the sum condition. Only after discovering his error does he generate additional pairs and test each one for the sum condition, thereby debugging his incomplete procedure.

D7's learning. D7, provided only with exercises to work, remembers (1 to 2) that sums and products have something to do with the process but applies the rule backwards (2), adding and multiplying the two coefficients in the polynomial and getting a wrong answer for the first problem. Apparently checking the answer (3 to 5) by the inverse operation of multiplying, he then (6) arrives at the right answer, which he checks by multiplication (7 to 11). He is now able to state the production rule (14 to 16).

D7's protocol shows that he was able to learn factoring—without being provided with any worked-out examples—simply by applying his knowledge (a) that factoring is the inverse of multiplication and (b) that he must make use, somehow, of sums and products of factors. Multiplying out the erroneous factors that represent his attempted solution of the first problem, he rediscovers the rules he should apply.

Progression in learning. These two protocols are not unrepresentative of the students' behavior in both conditions at the beginning of the session. An efficient procedure would be to factor the constant term of the quadratic first and then to find a pair of factors that sum to the coefficient of the linear term. Initially, however, most students (5 of 10 in E condition and 7 of 9 in D condition) look first for two numbers that add to the coefficient of the linear term. This preference may reflect the fact that, on scanning the quadratic from left to right, they first notice the linear term, derived from the factor sum, and then the constant term, derived from the product. By the end of the session, all those in the D group and all but one in the E group are using the more efficient procedure of satisfying the product condition before the sum condition.

Often, students do not initially check whether the numbers they have found to satisfy one condition also satisfy the second condition (4 of 10 E students and 5 of 10 D students fail to make such a test on the early exercises). As we have seen, E1 makes this error twice on the first three exercises and then notices his mistakes and corrects them. The error is usually noticed when the student checks his or her answer by multiplying the factors (E1, Lines 10 to 12).

If the student does not check the answer, the failure to satisfy the second condition may go unnoticed. Thus, the protocol of D4 on the second training problem of Part I reads:

> $x^2 + 7x + 6$, equals . . . [pause] That's 3, 4. By adding them, we get the coefficient of the second term. So I chose 3 and 4. $3 + 4 = 7$.

In this case, the error is not corrected. Such errors, in which one of the conditions is violated, are rare in the latter parts of the session.

In the earlier parts of the session, students frequently check their answers explicitly by multiplying out the two factors to obtain the original quadratic expression. Later, as they become skilled and accurate, this check, if made, is sometimes not verbalized. Two examples illustrate this difference among participants when they are solving some of the later problems. On Exercise 7 of Part III, D3 says only:

> $x^2 + 7x + 10 = (x + 5)(x + 2)$

The protocol of E3 on the same problem reads:

> $x^2 + 7x + 10$. 10 can be factored into 2 times 5. 2 plus 5 equals 7, which is the linear term's coefficient. So this problem equals $(x + 2)$ times $(x + 5)$.

Through all 13 Part III exercises, both students' protocols were essentially identical to their Exercise 7 protocols (with the appropriate numbers inserted in the grammatical "slots"). There is no reason to suppose the one had a better understanding of the process than the other; both were able to state the rule quite correctly and clearly. One was simply more taciturn than the other or had, perhaps, automated the solution process more completely.

All appeared to recognize that the constant term of the quadratic could be factored in more than one way, but they did not always find the pairs of factors systematically, and, in several cases, they took some time to realize that n can be factored into $1 \times n$. When they sought to satisfy the sum condition first, they often proceeded rather systematically on early problems. Later, on most problems, students mentioned only the pair of numbers that satisfied both conditions; the selection process was not verbalized except when they got stuck. A rare exception is represented by D2 on Exercise 5, Part III:

> [$x^2 + 11x - 12$] Equals x . . . Factor 12 into 3×4, $2 \times 6 = 12$. We don't get $11x$. Factor it into 1 and 12. x plus . . . x minus 1 times x plus 12.

On that same problem, D5 merely says:

> $x^2 + 11x - 12$. $x - 1$ times $x + 12$.

Differences between E and D conditions. From what has been said thus far, there do not appear to be large differences between students in the two conditions, although those in the D condition had to infer the factorization process without the help of worked-out examples. What differences appear are most clearly visible in the Part I problems. Those who are learning from examples typically examine several examples with some care and quite often refer back to them while working the first exercises. A clear case is E8:

> *Example 1:* $x^2 + 5x + 6 = (x + 2)(x + 3)$. This is factoring of the constant, 6, into the linear term, $5x$. It is, factor 6 into 2×3. Then the sum of the two factors is just the linear term: $2 + 3 = 5$, $2 \times 3 = 6$, just right. Consequently, plus x with this number, and then multiply . . . so it equals $(x + 2)(x + 3)$.
>
> *Example 2:* The same as the one above. Factor the constant, 6. The product of the two factors must equal this. The sum of the two factors must equal the linear term's coefficient, 7. It's just good to factor 6 into 1×6. So $1 + 6 = 7$ equals the coefficient of the linear term. Finally, multiplying the two factored polynomials. The answer is $(x + 1)(x + 6)$.
>
> *Exercise 1:* This is the same as the example. Factor 18 into two numbers, letting their sum equal 11. 18 can be factored into 2×9. $2 + 9 = 11$, which just equals the coefficient of the linear term, 11. So also factor x. Equals $(x + 2)(x + 9)$.

Notice (and this is common among E condition students) that E8 does not work mechanically from the examples but has clearly in mind the principle that factoring a quadratic is the inverse of multiplying two linear polynomials. This is especially evident from his comments on Example 2. For the D group, understanding this principle is essential to solving any of the problems. So, D3 says, on Training Problem 4 of Set 1:

> $x^2 + 7x + 12$. There are supposed to be two numbers in the parentheses. Their sum is 7, their product is 12. Equals $(x + 3)(x + 4)$.

D students who have not digested the principle from the definition of factoring quadratics have considerable difficulty with the initial training problems. Consider D2:

> *Training 1:* $x^2 + 5x + 6 = (x + \quad)(x + \quad)$. Plus 5. Hmm, wrong. Factor 5 into 2 . . . x. Factor 6 into 2×3. $3x$.
>
> *Training 2:* $x^2 + 7x + 6 = (x + \quad)(x + \quad)$. Factor 6 into . . . x^2 . . . x^2 . . . plus 2, x, hmm, wrong. Need one more x, $2x$. Since $2x$ equals

> x^2, then $3x$. $7x = 2x + \ldots$ Need one more x. $x^2 + 3x + 4x \ldots$ I don't know how to solve this problem.

He solves Problem 3 and falters on Problem 4 but henceforth seems to understand the principle and solves the problems successfully.

Knowledge of rule. By the end of Part I, almost all students (9 of 10 E students and 8 of 9 D students) can, in response to the experimenter's question, state fairly accurately the rule they are following in factoring but often in the form of an example rather than as a generalization. Thus D4 says, in answer to the question, "How do you factor?":

> Use multiplication. 2×3. If satisfied, see the linear term. See if $2 + 3$ equals the coefficient of the linear term. If yes, that's right. Thus it satisfies the problem. We got the coefficient of the linear term by addition.

In reply to the same question, E7 says:

> The sum of the two factors is the coefficient of the linear term. The product of two factors is the constant. All are the same.

Some answers refer explicitly to factoring as the inverse of multiplication. E4 says:

> Factoring equals multiplication formation—its inversion. Factoring is to get numbers by modification of the multiplication equations; to go back to get the initial ones before utilization of the multiplication process, i.e., before the calculation.

Some students in the E condition perhaps lean too heavily on the examples. Thus, E6 says, during Part II:

> For the problems in these two parts, read the examples first, then do the exercises. If I cannot solve the problems, I compare the exercises with the examples carefully.

It is not at all certain that this kind of direct comparison of exercises with examples will lead to understanding the principles. Most of the time, fortunately, students seem to try to extract the principles from the examples before they go on to the exercises. Some thought needs to be given to designing learning materials so that the latter strategy, rather than the former, is encouraged.

Protocols: Part II

On Part II, students compare the first and second examples and note that a negative sign in the linear term produces negative signs in both factors. Many (e.g., E10) are soon able to verbalize the rule:

> Well, mainly one has to see the linear term. If its coefficient has a plus sign, then the factors of the constant have plus signs . . . But if there is a minus sign in front of the linear term, then the two factors both have minus signs.

Some protocols provide evidence that the reason for the factor's being negative is understood. Thus, E5 says:

> The second one [the linear term] has a negative sign. So the latter two terms [the constants of the factors] must have negative signs . . . because minus, minus . . . These two both have negative signs.

In the D condition, the production for Part II is learned almost in the same way—by noting that the first and second problems differ only in the sign of the linear term. A little trial and error shows, if the linear term is negative, the signs of both factors should then be negative. When asked what rule he had found, one student (D7) replies:

> For example, $x^2 - 9x + 18$. $9x$ is negative, 18 is positive. In the parentheses they are all $(x - \ )$ . . . $(x - 3)$, $(x - 6)$. The signs in parentheses should all be minus.

They understand the reason for the negative factors, even when examples are not provided. D5, for example, is quite explicit, saying:

> According to the formula which I mentioned a moment ago, since the sum of these two numbers, 1 and 6, is the coefficient of the linear term, and the coefficient of the linear term is negative, so these two numbers should be negative. Negative times negative is positive.

In Part II, a few students give evidence that they are using the algebraic algorithm, which does not require them to distinguish different cases for the different arrangements of signs in the quadratic. This can be seen in E3's summary of the rules for Part II:

> I have found from here that one might factor the constant into these numbers. It might be factored into four pairs of numbers, 2 times 9, 3 times 6. Three pairs of numbers. 6 might be factored into 2 times 3.

> Factor the constant into a pair of numbers. If the sum of these two numbers equals the linear term's coefficient, these two numbers factored from the constant would be good. [Experimenter: "What rule have you found from this problem?"] It has been found from this problem that 6 can be factored into minus 2 times minus 3. The sum of minus 2 and minus 3 just equals the linear term's coefficient.

In general, the Part II problems were not difficult for students in either condition. When difficulties were encountered, they generally had to do with finding the correct pair of factors rather than with assigning the correct positive or negative signs to them. Nine E students and 7 D students were able to verbalize rules for handling the Part II problems.

Protocols: Part III

On the Part III problems, we again observe some trial and error. Students become aware that the assignment of signs to the factors depends on the pattern of signs in the quadratic; they gradually formulate the correct rules. Checking by multiplying the factors of the proposed answer reveals errors and allows the search to be successful.

Typically, they learned that the factors must have opposite signs before they understood which factor should have the positive sign and which the negative. D7 provides a characteristic first attempt at a (incorrect) rule:

> If the problem is (x^2 + −), then in the first parentheses it will be (x −); in the second parentheses it will be (x +).

After making and correcting several mistakes using this rule, the student corrects the rule:

> As far as the polynomial with successive subtractions [x^2 − −], its negative factor is larger than the positive factor. If the sign after x^2 is "+", and the next sign is "−" [x^2 + −], then also one factor is positive, the other is negative; but its positive factor is larger than the negative factor.

Table 3 gives the protocol of D6 on the first two exercises of Part III. In the first exercise, D6 attends to the constant term of the quadratic (Lines 1 to 3) but fails to notice the sign of the linear term (4 to 5). He then notices that Exercise 2 is identical to Exercise 1, except for the sign of the linear term (7 to 9). This observation leads to comparing the solutions of the two problems (10 to 12) and to the recognition (13 to 15) that his answer to Exercise 1 is wrong. Now the two answers are corrected (16 to 19), and the conditions for the coefficients of the linear terms are restated (20 to 27).

TABLE 3
Protocol of D6 on Part III of
Learning by Doing

1. [Working Exercise 1 in Part III] $x^2 + 5x - 6 = (x \quad 1)(x \quad 6)$
2. So it should be minus.
3. It should be $(x + 1)(x - 6)$.
4. Because -6 plus 1 equals 5
5. so it can match $5x$.
6. Positive multiplied by negative is negative too.
7. [Working Exercise 2] $x^2 - 5x - 6 = (x \quad 1)(x \quad 6)$
8. $-5x$ is negative
9. 6 is also negative
10. It should be $(x - 1)(x + 6)$.
11. Oh, it is right,
12. because $-5x$ is negative . . .
13. It is wrong,
14. it is positive.
15. I'm wrong in Problem 1, $x^2 + 5x - 6 = (x \quad 1)(x \quad 6)$.
16. It should be $(x - 1)(x + 6)$.
17. Problem 2, $x^2 - 5x - 6 = (x \quad 1)(x \quad 6)$
18. It should be reversed.
19. It should be $(x + 1)(x - 6)$,
20. because -6 plus 1 is equal to -5,
21. while this is $+5$.
22. $x^2 + 5x - 6 = (x \quad 1)(x \quad 6)$
23. so it can't match.
24. In the second problem, $x^2 - 5x - 6 = (x \quad 1)(x \quad 6)$
25. $+1$ plus -6 is equal to -5
26. It can match the number $-5x$
27. and $(+1) \times (-6)$ also equals -6 [the constant].

Note. For problems, see Appendix, 4B, Part III.

That the factors must be written down in some order causes some students to attend to this feature rather than to the relative sizes of the two factors; consequently, they arrive at a faulty generalization. Here is an example of such a faulty production (E10):

> From the above, it is clear that in the polynomial there is only one minus sign, and it is in front of the constant, then the front one of its two factors should be with a minus. And if there are minus signs in front of the first-order term and in front of the constant, then the factor which has the minus sign is the second one.

This error might be avoided if the pretraining included examples showing that when the order of factors to be multiplied is reversed, the same quadratic polynomial is produced.

Comparison of protocols of the students from Parts I and III shows that they gradually chunk the procedures they are learning into larger integrated units. So, in Part III, they typically verbalize their choice of signs but verbalize only the result of their choice of factors (E10):

> In Problem 4, because the polynomial has two minus signs, therefore the minus sign should be in the second factor. So $(x + 3)(x - 4)$.

In this protocol, as in many other Part III protocols, there is no evidence showing how the subject searched the factor pairs to find the one that also satisfied the sum condition. Some have also automated the assignment of signs in the same manner, verbalizing only the final answer.

They found the rules for Part III problems substantially harder to verbalize than the rules for the Parts I and II problems. Only 5 of the 9 D students and 6 of the 10 E students were able to provide reasonably accurate rules (sometimes in the form of examples) for the Part III problems, although in the posttest, all could solve these kinds of problems correctly. E3 provides a clear statement of the rule:

> For these problems, some have positive linear terms and negative constants. Thus, one factor is positive and another is negative. If the linear item is positive, the larger factor is positive and the smaller one is negative. If the linear term is negative, the larger factor is negative and the smaller one positive.

Discussion

The test results and protocols indicate that as long as the examples are appropriate and the problems are well arranged, most students master the skill of factoring quadratics quite well within a short time by working through examples and problems after being given knowledge that factoring is the inverse of multiplication. At the end, all these middle-school students had acquired about five productions, could use these productions to solve additional problems, and in most cases could verbalize them fairly accurately, if not elegantly.

The learning process. The process that students in the E condition use to acquire a production is approximately this: First they compare the polynomial with the factored expression to see what change has taken place. Based on their previous knowledge about multiplication of polynomials, and by trying to multiply $(x + c)(x + d)$, they notice that c and d add to the coefficient of x in the quadratic, and that c times d equals the constant. From this knowledge, they are able to compose the first production.

Similarly, from the Part II examples, they notice that a change in the sign of the linear term changes the sign of the factors. This enables them to state the conditions and action of the next production. The Part III examples lead to further discrimination of the four distinct cases on the basis of the pattern of signs in the quadratic.

Because the D condition provides no worked-out examples, this group of students must use their knowledge that factoring is the inverse of multiplication to solve the initial problems. D7's protocol, already discussed, gives a typical instance of earlier faltering but rapid progress as soon as one or two problems are solved.

What understanding is achieved? Because the product of the learning experience is a set of rules that most students can verbalize, should we conclude that the students have merely memorized these rules? We believe this is not a correct conclusion but that instead they understand the process—the learning has been meaningful. There are several reasons for this interpretation. To discuss them, we must say what we mean by "understanding."

We test whether someone understands knowledge by determining whether he or she can use it in appropriate ways. Knowledge can be understood shallowly or at great depth, the depth being measured by the range of tasks that can be performed with its help. Thus, one test of understanding factoring is to ask someone to solve factorization problems. Another test is to ask for the relation of factoring to multiplication. A third test is to ask for a derivation of the sum and product rules for the factors from the definition of factoring as the inverse of multiplication. A fourth test is to ask for a derivation of the pattern of signs in the factors from the pattern of signs in the quadratic. A fifth test is to ask for a verbal statement of the procedures for factoring. Even greater depths of understanding can be probed. Tasks can be proposed to see if the skill of factoring quadratics of the form $x^2 + ax + b$ can be transferred to quadratics of the form $x^2 + axy + by^2$ or $ax^2 + bx + c$. Or, in another direction, students might be asked to explain the relation between the factors of $x^2 + ax + b$ and the solution of the equation $x^2 + ax + b = 0$. As one measure that the semantics of factoring have been understood, students could be shown a rectangle with dimensions $(x + a)$ and $(x + b)$ and be asked to identify the subareas of the rectangle that correspond to the terms of the quadratic: x^2, ax, bx, and ac. At the level of a university course in abstract algebra, other questions, much deeper than these, can be posed.

In terms of this notion of understanding, it is clear that participants in this experiment had not merely acquired rote learning about factoring quadratics but had acquired an understanding of considerable depth—some, of course, to a greater depth than others.

The first evidence of their understanding is that they can do more than merely recite the rules; they also can and do apply them skillfully by making the appropriate perceptual discriminations and then carrying out the appropriate actions. They recognize when each rule is relevant, and then use it.

Second, the students demonstrate understanding by using the basic meaning of factoring to check their trial results by multiplying back. They have seen how the corresponding patterns of signs in the quadratic and its factors derive from the nature of the multiplication process for polynomials. They have also seen how the sum and product of the factors appear in the resulting quadratic. The D condition students have even demonstrated that they can acquire these skills by derivation from the basic definition, without the help of examples.

For these reasons, we must conclude that the students have learned a good deal about the semantic meaning of factoring, as well as the skill of doing it, and that it is from this knowledge that they generate their versions of the verbal rules governing it. Because the rules were never given to them explicitly, they certainly have not simply memorized them and recited them by rote.

For some of the students (but probably only a few), understanding was limited largely to being able to factor. However, as our account has shown, most of them also understood why the rules worked as they did. We did not probe deeper levels of understanding—how broadly the new skill would transfer, whether they could understand a geometric representation of the factoring of a quadratic, or whether, which is highly doubtful, they had acquired any of the understanding that would be expected in a course of abstract algebra. They did appear to understand most of the things that first-year algebra textbooks undertake to teach about factoring quadratics.

In sum, Experiment 1 demonstrated that students can learn how to factor quadratics by studying examples and/or by working problems and that they can do so relatively quickly and efficiently if care is taken to arrange the examples and problems so that they do not make too many errors of induction or require too much trial-and-error search.

ADDITIONAL EXPERIMENTS

The success of our initial experiment, completed in 1983, encouraged us to extend the inquiry in several directions. First, we wished to replicate the experiment on factoring quadratics, and include comparisons with control groups who were taught by lecture in the normal way. Second, we wished to generalize the result to other tasks. Because our main purpose in this article is to discuss the processes of learning from examples rather than the efficiency of this procedure under classroom conditions, we only summarize these experimental findings, leaving a detailed analysis to a later report.

With respect to replication, we obtained very similar findings on the factoring task in three other schools, and the results were favorable in comparison with control groups. As for new tasks, we constructed and tested materials for a task of manipulating exponents, a geometry task, a lesson in ratios and proportions, and three lessons in fractions. For all tasks, the students in the experimental condition were usually more successful, and usually took less time to learn, than the students in the control condition.

Replications of the Factoring Task

In School A, with 32 students in each condition, the average posttest score for the experimental students was 93.13% correct, and for the control students, 75.50% correct. The difference was significant at the .001 level by t test. In School B, with 39 students in the experimental condition and 38 in the control condition, the average scores were 97.23% and 95.08%, respectively. Here the difference is not significant. In School A, the control students took slightly (but not significantly) longer to learn than the experimental students. In School B, there was no difference in learning times. (The average time in School A was about 40 min, and in School B about 30 min.)

Exponent Task

In the experiments with exponents, students learned the three formulas: $a^m \times a^n = a^{m+n}$; $(a^m)^n = a^{m \times n}$; and $(ab)^m = a^m \times b^m$. On problems involving the first formula, 32 of 36 students (89%) scored 80% or higher; on the second formula, all 36 students scored 80% or higher; on the third formula, 21 of 35 students scored 80% or higher, and all 35 scored 60% or higher.

These results were replicated in 1985 in comparison with a control group. The experimental group scored higher than the controls on all three laws of exponents (significantly higher at the .05 and .001 levels on the second and third laws) and required only 180 min for learning, as compared with 270 min for the controls. The average percentage scores of the experimental students for the three laws of exponents were 93.1, 95.4, and 90.2 and the average score for tasks involving several of the laws together was 88 (as compared with 73.1 for the controls).

Geometry Task

The geometry task involved proving properties of triangles, parallelograms, and their components. The experiment was run in 1984 in four different schools, improvements being introduced into the training materials on the basis of the experience gained in the first two schools. The average scores in the four schools were 76.9%, 77.9%, 89.1% and 83.5%, respectively. In the last two schools, the experimental groups had slightly higher (but not significantly higher) scores than the control groups (89.1% vs. 85.2% in School M, 83.5% vs. 83.2% in School N). Significantly more students acquired the skill in the experimental groups (94.6% vs. 88% in School M, 90.6% vs. 85% in School N). Moreover, the average learning times required by the experimental groups were substantially shorter than those required by the control groups (50 min vs. 75 min in School M, 50 min vs. 70 min in School N).

In a 1985 replication of the geometry experiment in one school, the average score of the experimental group was 89%, and the average score of the control group was 83%, a nonsignificant difference at the .05 level. Both groups required the same class learning time (50 min), but the control group spent an additional 20 min doing homework.

Ratios and Fractions

The 1985 results with lessons on ratios, proportions, and fractions were similar to those of the other experiments. The experimental groups performed as well as, or slightly better than, the control groups on all tasks, and required substantially less time to learn the tasks of manipulating fractions.

Results of Retests

In comparing different instructional methods, the relative durability of the skills attained by the one method or the other is of the greatest importance. Retention also bears on the question of whether the learning was rote or meaningful, for Katona (1940) showed that material learned with understanding is better retained than material learned by rote.

Retention after 1 year was tested for two of the tasks already discussed: factoring and geometry. A class that learned to factor quadratics in 1984, with an average score of 97.2%, obtained an average score of 96.0% when retested in 1985. The control class, which had an average score of 95.0% in 1984, had an average score in 1985 of 92.1%. Thus, retention was good for both groups but was perhaps slightly better for the experimental group.

With respect to geometry, experimental and control classes that scored 89.1 and 85.2, respectively, in 1984, scored 96.3% and 86.7%, respectively, in 1985—showing some advantage for the experimental group.

All these additional experiments further demonstrate the feasibility of teaching students through worked-out examples and carefully designed sequences of problems. Although the differences in success between these methods and traditional methods of instruction were generally modest in magnitude, they were positive, and they were generally achieved with a savings—sometimes substantial—in time.

The geometry experiment showed the importance of designing the instructional materials carefully. By redesigning the materials, we were able to enhance students' learning performance and, in particular, to enable most of the weaker students to acquire the ability to solve the problems.

As we pointed out earlier, our analysis of the protocols of students in the first experiment gives us strong reason to believe that the students are learning in a meaningful fashion and not simply by rote. This conclusion is borne out by the high levels of retention of the skills over 1 year.

Experiment With a Full Course

Within the past 2 years, materials have been completed for teaching—by the learning-from-examples method—the standard curriculum in the Chinese middle schools of 2 years of algebra and 1 year of geometry; these materials have been used—following the method—with a Beijing middle-school class judged to be of "average" ability. In 2 years, most students in the class have completed the entire 3-year curriculum. The average score of these students on the standard mathematics tests given to children all over China (which they took in June 1987) was just slightly higher than the average score of three other classes of students in the same school who started 1 year earlier and completed the regular curriculum, taught by standard methods, in 3 years. Slightly more than 87% of the experimental students, and 81% of the students in the three regular classes, received scores above 60%—the grade regarded as "passing."

Even if the evidence becomes clear that a time saving of one third has actually been achieved, the findings will need to be interpreted with caution. It is one thing to conclude that children can learn from examples and do so efficiently. It is quite another to conclude that the time saving was *caused* by the method of instruction used.

This is especially so because other experiments in algebra instruction in Chinese schools have shown comparable savings. For example, there are Professor Zhongheng Lu's (1983, 1984, 1986) experiments with the technique of self-study guided textbooks, and similar guided self-study texts have been compiled in Shanghai and Guandong. Experiments with these texts have produced quite good results. This shows that the "let a hundred flowers bloom" policy in educational experimentation can produce fruitful scientific outcomes.

What these methods, including ours, have in common is that the primary responsibility for learning is placed on the student, who spends his time actively reading and solving problems, and not passively attending to an instructor. Of course, there is also the possibility that the effectiveness of both methods stems from a "Hawthorne effect," the students being motivated by the knowledge that they are the subjects of special attention and that they are using novel methods of learning.

All these possibilities are quite compatible. There is ample evidence that students learn only when they are attending to the task. The motivation for appropriate attention may come from many sources. One important one, evidently, is the availability of materials that allow students to adopt active strategies in their learning, so that a maximum of time is devoted to relevant activities. We make no claim that learning from examples is the sole route to this goal, although the evidence thus far suggests that it is a very effective one.

CONCLUSIONS

Our experiments provide substantial evidence for the possibility of teaching several different mathematics skills by presenting students with carefully chosen sequences of worked-out examples and problems, without lectures or other direct instruction. In learning by these methods, the students were at least as successful as, and sometimes more successful than, students learning by conventional methods, and in most cases they learned in a shorter time.

Although the students were usually able to state the rules they had learned, they had not simply memorized these rules, for they were able to recognize when the rules applied to a problem and then to apply them. Moreover, protocol evidence shows that the students understood the semantic meanings of the rules in terms of their derivation from the fundamental definition of factoring. Hence we may say that the learning that took place was learning with understanding and was not merely rote learning.

Materials for teaching 3 years of middle-school mathematics by the learning-from-examples method have now been developed and have been tested, thus far very successfully, with a middle-school class in Beijing. As next steps in the study of learning from examples and in its practical applicability to school instruction, experiments are needed on a wider range of school topics, and further tests are needed of the retention and transfer of skills acquired.

Because the processes of learning from examples are well adapted to computer-aided instruction, the method has considerable potential for wide use. Whether computers are used, or simple paper-and-pencil materials, the methods described here give promise of freeing considerable teacher time for working with individual students who are having special difficulties.

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APPENDIX

Materials for Experiment 1

Production System

As a basis for preparing the training materials, a detailed analysis was made of the factoring task, and a production system was constructed that was powerful enough to perform the task successfully. This production system is assumed to represent (approximately) the set of skills that students must acquire to do the task. It served as a guide for designing the teaching materials. It consists of seven productions:

1. If the goal is to factor an expression like $(x^2 + ax + b)$,

   $\rightarrow$

   x^2 is factored as $x \cdot x$.

2. If the goal is to factor a quadratic expression,

   $\rightarrow$

   find all pairs of positive integers whose product equals the absolute value of the constant term of the expression.

3. If the constant term and the coefficient of the linear term are both positive,

   $\rightarrow$

   select a pair of factors whose sum equals the absolute value of the linear term's coefficient.

4. If the constant term is positive and the coefficient of the linear term negative,

   $\rightarrow$

   select a pair of factors whose sum is equal to the absolute value of the linear term's coefficient and take their negatives.

5. If the constant term is negative and the coefficient of the linear term positive,

   →

   select a pair of factors whose difference is equal to the linear term's coefficient and set the smaller factor negative.

6. If both the constant term and the coefficient of the linear term are negative,

   →

   select a pair of factors whose difference is equal to the linear term's coefficient and set the larger factor negative.

7. If $\{c, d\}$ is the pair of factors that has been selected,

   →

   write $x^2 + ax + b = (x + c)(x + d)$.

1. First Pretest

Fill in the blanks: The correct answers are shown in angle brackets, < >. Multiple choice alternatives shown to the students are given in parentheses.

1. *If:* the goal is to factor a 3-termed quadratic polynomial, $x^2 + 5x + 6$, with the coefficient of x^2 equal to 1,
Then: x^2 factors as <$x \cdot x$>, the constant term is the <product> (product, sum, quotient) of the <two> factors, and the coefficient of the linear term is the <sum> (sum, product, quotient) of these <two> factors.

2. *If:* the constant term is positive,
Then: the two factors of the constant term have <the same> (different, the same, positive, negative) signs.

3. *If:* the constant term is positive, and the linear term is also positive,
Then: the two factors of the constant term are <positive> (different, the same, positive, negative).

4. *If:* the constant term is positive, and the linear term is negative,
Then: the two factors of the constant term are <negative> (different, the same, positive, negative).

5. *If:* the constant term is negative, and the linear term is positive (e.g., $x^2 + 4x - 12$),

Then: the two factors of the constant term are <opposite> in sign, and the positive factor is <larger> in size than the negative one.

6. *If:* the constant term is negative, and the linear term is negative (e.g., $x^2 - 4x - 12$),
Then: the two factors of the constant term are <opposite> in sign, and the negative factor is <larger> in size than the positive one.

2. Review Knowledge

Students in both conditions studied the following materials:

Fill in the blanks:

Example: 8 can be expressed as the product of the two numbers, 2×4, and the sum of these two numbers is 6.
Problems: 8 can also be expressed as the product of two numbers, 1×8, and the sum of these two numbers is ____.
6 can be expressed as the product of two numbers, ____, and the sum of these two numbers is ____.
6 can be expressed as the product of two numbers, ____, and the sum of these two numbers is ____.
Example: In $x^2 + 5x + 6$, x^2 is the quadratic term, $5x$ is the linear term, 6 is the constant term.
Problems: In $x^2 + 8x + 12$, x^2 is ____, $5x$ is ____, 12 is ____.
In $x^2 + 13x + 12$, 12 is ____, x^2 is ____, $13x$ is ____.

Using the rules of repeated multiplication, work the problems below.

Example: $(x + 2)(x + 4) = x^2 + 4x + 2x + 8 = x^2 + 6x + 8$.
Problem: $(x + 3)(x + 5) = x^2 + \_\_ + \_\_ + 15 = x^2 + \_\_ + 15$.

Inverse process: $x^2 + 6x + 8 = (x + 2)(x + 4)$.
$x^2 + 6x + 8$ can be expressed as the product of $(x + 2)$ and $(x + 4)$.
The components are called the factors of the quadratic.

3. Second Pretest

Factor the following:

(1) $x^2 + 7x - 18 =$
(2) $x^2 - 2x - 8 =$

(3) $x^2 + x - 20 =$
(4) $x^2 - 5x - 36 =$
(5) $x^2 - 9x + 14 =$

4A. Learning From Examples

Part I

Examples: (1) $x^2 + 5x + 6 = (x + 2)(x + 3)$
(2) $x^2 + 7x + 6 = (x + 1)(x + 6)$
(3) $x^2 + 8x + 12 = (x + 2)(x + 6)$
(4) $x^2 + 7x + 12 = (x + 3)(x + 4)$
(5) $x^2 + 13x + 12 = (x + 1)(x = 12)$

Exercises: (1) $x^2 + 11x + 18 = (\quad)(\quad)$
(2) $x^2 + 9x + 18 = (\quad)(\quad)$
(3) $x^2 + 19x + 18 = (\quad)(\quad)$

Part II

Examples: (1) $x^2 + 5x + 6 = (x + 2)(x + 3)$
(2) $x^2 - 5x + 6 = (x - 2)(x - 3)$
(3) $x^2 + 7x + 6 = (x + 1)(x + 6)$
(4) $x^2 - 7x + 6 = (x - 1)(x - 6)$

Exercises: (1) $x^2 + 9x + 18 = (\quad)(\quad)$
(2) $x^2 - 9x + 18 = (\quad)(\quad)$
(3) $x^2 - 11x + 18 = (\quad)(\quad)$
(4) $x^2 + 11x + 15 = (\quad)(\quad)$

Part III

Examples: (1) $x^2 + 5x - 6 = (x - 1)(x + 6)$
(2) $x^2 - 5x - 6 = (x + 1)(x - 6)$
(3) $x^2 + x - 6 = (x - 2)(x + 3)$
(4) $x^2 - x - 6 = (x + 2)(x - 3)$

Exercises: (1) $x^2 + 4x - 12 = (\quad)(\quad)$
(2) $x^2 - 4x - 12 = (\quad)(\quad)$
(3) $x^2 + x - 12 = (\quad)(\quad)$
(4) $x^2 - x - 12 = (\quad)(\quad)$
(5) $x^2 + 11x - 12 = (\quad)(\quad)$
(6) $x^2 - 11x - 12 = (\quad)(\quad)$
(7) $x^2 + 7x + 10 = (\quad)(\quad)$
(8) $x^2 - 6x + 8 = (\quad)(\quad)$
(9) $x^2 + 7x - 18 = (\quad)(\quad)$

(10) $x^2 - 2x - 8 = (\quad)(\quad)$
(11) $x^2 + x - 20 = (\quad)(\quad)$
(12) $x^2 - 5x - 36 = (\quad)(\quad)$
(13) $x^2 - 9x + 14 = (\quad)(\quad)$

4B. Learning by Doing

Part I

Problems: Identical with examples for learning-from-examples section except that right-hand sides of equations are shown as:

$$= (x + \_\_)\,(x + \_\_)$$

Exercises: Identical with exercises for learning-from-examples section.

Part II

Problems: As before, except that right-hand sides are shown as:

(1) $= (x \_\_ 2)(x \_\_ 3)$
(2) $= (x \_\_ 2)(x \_\_ 3)$
(3) $= (x \_\_ 1)(x \_\_ 6)$
(4) $= (x \_\_ 1)(x \_\_ 6)$

Exercises: As before.

Part III

Problems: As before, except that right-hand sides are shown as:

(1) $= (x \_\_ 1)(x \_\_ 6)$
(2) $= (x \_\_ 1)(x \_\_ 6)$
(3) $= (x \_\_ 2)(x \_\_ 3)$
(4) $= (x \_\_ 2)(x \_\_ 3)$

Exercises: As before.